Published by Institute of Physics Publishing for SISSA/ISAS

Received: *July 25, 2001*
Revised: *January 8, 2002*
Accepted: *March 6, 2002*

Noncommutative solitons on Kähler manifolds

Marcus Spradlin and Anastasia Volovich*

Jefferson Physical Laboratory, Harvard University
Cambridge MA 02138, USA
E-mail: spradlin@physics.harvard.edu, nastya@physics.harvard.edu

Abstract: We construct a new class of scalar noncommutative multi-solitons on an arbitrary Kähler manifold by using Berezin's geometric approach to quantization and its generalization to deformation quantization. We analyze the stability condition which arises from the leading $1/\hbar$ correction to the soliton energy and for homogeneous Kähler manifolds obtain that the stable solitons are given in terms of generalized coherent states. We apply this general formalism to a number of examples, which include the sphere, hyperbolic plane, torus and general symmetric bounded domains. As a general feature we notice that on homogeneous manifolds of positive curvature, solitons tend to attract each other, while if the curvature is negative they will repel each other. Applications of these results are discussed.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry.

*On leave from the L.D. Landau Institute for Theoretical Physics, Moscow, Russia.

Contents

1. Introduction

Starting from the celebrated paper of Gopakumar, Minwalla and Strominger [1], there has been a lot of interest in studying solitonic solutions of noncommutative field theory. Indeed, noncommutative geometry has arisen in at least three distinct but closely related contexts in string theory. Witten's open string field theory [2] formulates the interaction of bosonic open strings in the language of noncommutative geometry. Compactification of matrix theory on the noncommutative torus [3] was argued to correspond to supergravity with constant background three-form tensor field. More generally, it has been realized [4, 5] that noncommutative gauge theory arises in the worldvolume theory on D-branes in the presence of a constant background B field in string theory.

Although Derrick's theorem forbids solitons in ordinary 2+1-dimensional scalar field theory, solitons in noncommutative scalar field theory on the plane were constructed in [1] (see also [6, 7, 8] for reviews). It was soon realized that noncommutative solitons represent D-branes in string field theory with a background B field turned on [9], and this has allowed confirmation of some of Sen's conjectures [10] regarding tachyon condensation in string field theory. Other recent work on noncommutative solitons include [11, 12, 13] for scalar solitons and [14] for solitons in noncommutative gauge theories. Gauge theory solitons have been studied on the sphere in [15], and on the torus in [16]. Instanton solutions have been studied in [17].

In this paper we develop a general approach to the study of solitons in noncommutative scalar field theory on any Kähler manifold. We define a noncommutative field theory using the Berezin $\star$ product [18] and its generalization to deformation quantization [19, 20] and construct static solutions of the theory.

The basic idea [1] behind the study of noncommutative solitons is familiar by now. One begins by exploiting an isomorphism between the algebra of functions with the noncommutative $\star$ product and the algebra of operators on some Hilbert space. For example, the algebra of functions on the plane with the Moyal $\star$ product can be represented as the algebra of operators on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. In this sense the *coordinate* space on which the noncommutative field theory is defined (i.e., the plane) is identified as the *phase* space of an auxiliary quantum mechanics (i.e., a particle on the line). Noncommutative field theories defined in this way on compact manifolds necessarily have a finite number of degrees of freedom, since the corresponding Hilbert space is finite dimensional. It is not yet clear whether it is possible to define noncommutative field theories with an infinite number of degrees of freedom on curved compact spaces.

In a limit which corresponds roughly to infinite noncommutativity, the potential term in the action dominates over the spatial part of the kinetic term (henceforward referred to simply as the kinetic term, since we will only consider static solutions). Static solutions to the approximate equation of motion $V'(\phi) = 0$ may be constructed using projectors in the relevant operator algebra. Let us emphasize that we will also construct solitons based on functions ϕ which satisfy $\phi \star \phi = \phi$ in the formal sense of deformation quantization. Since there is no Hilbert space in this case, we will refer to these as 'projectors' rather than 'projection operators.'[1]

One typically analyzes the stability of these solutions by doing perturbation theory around infinite noncommutativity. The space of rank k projection operators is isomorphic to the grassmannian $\mathrm{Gr}(k, N)$ of complex k-planes in $\mathbb{C}^N$, where N is the dimension of the Hilbert space. The leading correction to the energy comes from the kinetic term, which introduces an effective potential on $\mathrm{Gr}(k, N)$. It was shown in [1, 12, 13] that on the plane, any projection operator whose image is spanned by coherent states minimizes the kinetic term, thus there is a moduli space for solitons.[2]

Note that in geometric quantization on a compact manifold, the Hilbert space is always finite dimensional, so that the corresponding space of projection operators $\mathrm{Gr}(k, N)$ is also compact. Therefore the kinetic term introduces a potential on $\mathrm{Gr}(k, N)$ which is bounded above and below. Depending on one's application, it may therefore not even be necessary

[1]Projectors in a subalgebra of the string field theory algebra have recently been shown to play an important role in the construction of D-branes in split string field theory [21].

[2]Actually, the moduli space can be quite intricate when two or more separated coherent state solitons are brought together [12]. This is discussed more below.

to consider minimizing that first perturbation to the energy. By contrast on a noncompact manifold, the kinetic term is generally unbounded, so that the perturbative expansion makes no sense unless one is careful to analyze the contribution from the kinetic term.

On a completely general manifold, the kinetic term has no symmetry and thus one would not even expect a moduli space for a single soliton. In this paper we show that on a homogeneous Kähler manifold (of the type classified in section 5.4), there is a moduli space for a single soliton represented by a projection operator onto a generalized coherent state.

The organization of the paper is as follows. In section 2 we review Berezin's approach to the quantization of Kähler manifolds and its generalization to deformation quantization. In section 3 we describe the noncommutative scalar field theory that we will be studying and construct its multi-soliton solutions as projectors in the corresponding algebra of functions. In section 4 we analyze the stability condition for these solitons, which we are able to solve for homogeneous Kähler manifolds, where stable solitons are given in terms of generalized coherent states. Sections 5 and 6 are devoted to examples. In section 5.1 we connect our general formalism to the very well-studied case of the plane. The $\star$ product that we are using reduces on the plane to the normal ordered operator product, which is equivalent to the more familiar Moyal product after a nonlocal field redefinition. In sections 5.1–5.3 we study general symmetric bounded domains, concentrating in great detail on the sphere and hyperbolic plane. Among other things we show that that on a positive curvature manifold solitons will tend to attract each other, while on a negative curvature manifold they will repel each other (although a discussion of why this might not generalize to non-homogeneous manifolds is given in section 7). We also show explicitly how the fuzzy versions of these manifolds exactly map to Berezin $\star$ product. Section 6 is devoted to the torus, where the breaking of translational symmetry to $\mathbb{Z}_N \times \mathbb{Z}_N$ plays a crucial role. We demonstrate the equivalence of the Berezin $\star$ product to the fuzzy torus, and analyze a proposal for the definition of a kinetic term. We conclude the paper in section 7 with a discussion of the results and their possible applications.

2. Quantization of Kähler manifolds

In this section we briefly review Berezin's approach [18] to the quantization of Kähler manifolds using generalized coherent states and its generalization to deformation quantization [19, 20].[3]

Let us recall the problem of quantization of a Poisson manifold $\mathcal{M}$. Let A be the Lie algebra of smooth functions on $\mathcal{M}$ with the Poisson bracket

$$\{f, g\} = \omega^{ij} \partial_i f \partial_j g \,, \tag{2.1}$$

where ω is the Poisson structure on $\mathcal{M}$. Quantization is defined as a family $\mathcal{A}_\hbar$ of deformations of the algebra $\mathcal{A}$, where $\hbar$ is a parameter which takes values in some subset of

[3]Recently this approach has also been used in [22] for quantization of the horizon in de Sitter space.

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positive real numbers, such that $\mathcal{A}_\hbar$ reduces to $\mathcal{A}$ in the limit $\hbar \to 0$ in the following sense:

$$\lim_{\hbar\to 0} f \star_\hbar g = fg\,, \qquad \lim_{\hbar\to 0} \frac{1}{\hbar}(f\star_\hbar g - g\star_\hbar f) = i\{f,g\}, \tag{2.2}$$

where $\star_\hbar$ is the multiplication operator in $\mathcal{A}_\hbar$. (Henceforward we will drop the $\hbar$ subscript on $\star$ for notational simplicity.)

In geometric quantization one associates to $\mathcal{M}$ a family of Hilbert spaces $\mathcal{H}_\hbar$, with a correspondence between real-valued functions in $\mathcal{A}_\hbar$ (i.e, observables) and hermitean operators $\mathcal{O}_f$ on $\mathcal{H}_h$ such that $\mathcal{O}_{f\star g} = \mathcal{O}_f\mathcal{O}_g$. The more general concept is deformation quantization [19, 20] where the algebra of functions is not necessarily representable as operators on any Hilbert space.

In geometric quantization (for instance see [23] for a review), the correspondence between functions f and operators $\mathcal{O}_f$ requires the introduction of a symplectic potential θ such that $\omega = d\theta$. Of course θ will in general not be globally defined, so the construction depends on a local coordinate chart. Since the action of a quantum operator on a wavefunction $|\psi\rangle$ should be independent of the local chart, the Hilbert space consists not of functions on $\mathcal{M}$ but rather sections of a complex line bundle L over $\mathcal{M}$. Additionally one requires a "polarization" which roughly speaking is a splitting of the coordinates on the phase space $\mathcal{M}$ into those which are q's (coordinates) and those which are p's (momenta). The Hilbert space is then restricted to those wavefunctions which depend only on the q's. On complex manifolds one takes the complex structure as the polarization; then the Hilbert space $\mathcal{H}$ is spanned by the holomorphic sections of L.

Berezin's approach to geometric quantization [18] is based on the use of generalized coherent states (see [24] for a review on coherent states) and will be well-suited to the study of noncommutative solitons. His construction is also local, and therefore best suited for manifolds for which there is an open dense subset which can be covered with a single chart. A global construction may be found in [25], but will not be necessary here.

Let (M,ω) be a Kähler manifold, so that in local coordinates the metric and the Kähler form are

$$ds^2 = g_{i\bar{k}}dz^i dz^{\bar{k}}\,, \qquad \omega = g_{i\bar{k}}dz^i\wedge dz^{\bar{k}}\,, \tag{2.3}$$

where $g_{i\bar{k}} = \partial_i\partial_{\bar{k}}K$ and K is the Kähler potential.

Berezin considered a complex line bundle L over $\mathcal{M}$ with a fiber metric $e^{-\frac{1}{\hbar}K(z,\bar{z})}$.[4] The Hilbert space $\mathcal{H}_\hbar$ therefore consists of holomorphic sections of L with the inner product

$$\langle f|g\rangle = c(\hbar)\int \overline{f(z)}g(z)e^{-\frac{1}{\hbar}K(z,\bar{z})}d\mu(z,\bar{z})\,, \tag{2.4}$$

where $c(\hbar)$ is a normalization constant to be chosen shortly and $d\mu$ is the measure

$$d\mu(z,\bar{z}) = \det||g_{i\bar{k}}||\prod\frac{dz^k\wedge d\bar{z}^k}{2\pi i}\,. \tag{2.5}$$

[4] We will shortly see that for compact manifolds, a quantization condition will relate the dimension of $\mathcal{H}_\hbar$ to $\hbar$ and the volume of $\mathcal{M}$.

Let $\{f_k\}$ be a basis of holomorphic sections orthonormal with respect to (2.4). The basis may be finite or infinite, depending on whether $\mathcal{M}$ is compact. The Bergman kernel [26] is defined by

$$B_\hbar(z,\bar{z}) \equiv \sum_k f_k(z)\overline{f_k(z)}\,. \tag{2.6}$$

It projects all measurable and square integrable functions onto $\mathcal{H}_\hbar$. One can prove that it is independent of basis. The holomorphic sections defined for $v \in \mathcal{M}$ by

$$|v\rangle \equiv B_\hbar(\cdot,\bar{v}) \tag{2.7}$$

are called generalized coherent states[5] and form an overcomplete system in $\mathcal{H}_\hbar$. We will use $\langle v| = |v\rangle^\dagger$ to denote the antiholomorphic section $B_\hbar(v,\bar{z})$. We choose the constant $c(\hbar)$ so that the resolution of the identity reads

$$\mathbf{1} = c(\hbar)\int |v\rangle\langle v| e^{-\frac{1}{\hbar}K(v,\bar{v})} d\mu(v,\bar{v})\,. \tag{2.8}$$

Note that for any holomorphic section $f(z)$ we have

$$\langle v|f\rangle = f(v)\,. \tag{2.9}$$

For each bounded operator $\hat{O}$ on $\mathcal{H}_\hbar$ we can define its symbol with respect to the system $|v\rangle$ as

$$O(z,\bar{v}) = \frac{\langle z|\hat{O}|v\rangle}{\langle z|v\rangle}\,. \tag{2.10}$$

Using (2.8) we can write a formula for the action of $\hat{O}$ on an arbitrary section $f(z)$:

$$\begin{aligned}(\hat{O}f)(z) = \langle z|\hat{O}|f\rangle &= c(\hbar)\int O(z,\bar{v})\langle z|v\rangle\langle v|f\rangle e^{-\frac{1}{\hbar}K(v,\bar{v})} d\mu(v,\bar{v}) \\ &= c(\hbar)\int O(z,\bar{v}) B_\hbar(z,\bar{v}) f(v) e^{-\frac{1}{\hbar}K(v,\bar{v})} d\mu(v,\bar{v})\,.\end{aligned} \tag{2.11}$$

Functions in $\mathcal{A}_\hbar$ are interpreted as symbols of operators on $\mathcal{H}_\hbar$. The product of operators corresponds to the star product of symbols. Multiplication in $\mathcal{A}_\hbar$ is defined as

$$(O_1 \star O_2)(z,\bar{z}) = c(\hbar)\int O_1(z,\bar{v}) O_2(v,\bar{z}) \frac{e_\hbar(z,\bar{v}) e_\hbar(v,\bar{z})}{e_\hbar(z,\bar{z})} e^{\frac{1}{\hbar}\Phi(z,\bar{z}|v,\bar{v})} d\mu(v,\bar{v})\,, \tag{2.12}$$

where we have defined

$$e_\hbar(z,\bar{z}) \equiv B_\hbar(z,\bar{z}) e^{-\frac{1}{\hbar}K(z,\bar{z})} \tag{2.13}$$

and $\Phi(z,\bar{z}|v,\bar{v})$ is the Calabi diastatic function

$$\Phi(z,\bar{z}|v,\bar{v}) = K(z,\bar{v}) + K(v,\bar{z}) - K(z,\bar{z}) - K(v,\bar{v})\,, \tag{2.14}$$

which is manifestly invariant under Kähler transformations and therefore globally defined.

[5] In the global formulation, coherent states are parametrized not by $\mathcal{M}$ but rather by L_0, which is L minus the image of the zero section [27].

The trace of a symbol is defined by

$$\mathrm{Tr}[O] = c(\hbar) \int O(z, \bar{z}) e_\hbar(z, \bar{z}) d\mu(z, \bar{z}) \tag{2.15}$$

and is cyclically invariant. Substituting the identity operator, whose symbol is 1, into (2.15) we find

$$\dim \mathcal{H}_\hbar = c(\hbar) \int e_\hbar(z, \bar{z}) d\mu(z, \bar{z}) \,. \tag{2.16}$$

Other properties of the $\star$ product are summarized in the appendix.

Berezin was only able to prove that his quantization procedure satisfies the correspondence principle (2.2) under very restrictive assumptions on the geometry of $\mathcal{M}$ [18]. This quantization works for the case when $\mathcal{M}$ is a homogeneous Kähler manifold and L is a homogeneous bundle (later it was generalized to generalized flag varieties [28]). Under these assumptions $e_\hbar = 1$, and thus most of the formulas simplify greatly.

Nevertheless, the formula (2.12) defines a $\star$ product on any Kähler manifold in the formal sense of deformation quantization [20]. That is, although it appears that the formulas (2.11) and (2.12) only make sense for functions which admit an analytic continuation $f(z, \bar{v})$ to all of $\mathcal{M} \times \mathcal{M}$, in deformation quantization these integrals are treated as formal power series in $\hbar$. Then the star product does not require such an analytic continuation but depends only on the derivatives (of all orders) of $f(z, \bar{z})$ evaluated on the diagonal. Deformation quantization therefore works for all smooth functions, and the star product reduces to Berezin's formula (2.12) for those functions which do admit an analytic continuation.

3. Noncommutative solitons and projectors

We consider a noncommutative scalar field theory on $\mathbb{R} \times \mathcal{M}$, where $\mathcal{M}$ is an arbitrary Kähler manifold of complex dimension n, with the action

$$S = \int dt \int d\mu(z, \bar{z}) \left(\frac{1}{2} \partial_t \phi \star \partial_t \phi + \phi \star \triangle \phi - m^2 V(\phi)_\star \right) . \tag{3.1}$$

Here $\star$ is as defined in (2.12) and $\triangle \equiv g^{i\bar{j}} \partial_i \partial_{\bar{j}}$ is the scalar laplacian on $\mathcal{M}$. The subscript on the potential V indicates that it is evaluated with the star product. We will assume that V is bounded below, to ensure that stable solutions exist.

Note that unlike the perhaps more familiar case of the Moyal product on the plane, the star product cannot be omitted from the first term in (3.1). However in this paper we are interested in static solutions of (3.1), so this fact will play no role and our task will simply be to find minima of the energy functional

$$E[\phi] \equiv \hbar^{n-1} \int d\mu(z, \bar{z}) \left(-\phi \star \triangle \phi + m^2 \hbar V(\phi)_\star \right) . \tag{3.2}$$

In writing this formula we have made use of the fact that by rescaling the coordinates and size of $\mathcal{M}$ one can formally eliminate $\hbar$ from the star product (2.12).[6]

[6]Instead, it will depend on the scale of curvature of $\mathcal{M}$ (measured in units of $\hbar$), or equivalently on the dimension of the Hilbert space, if $\mathcal{M}$ is compact.

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One of the fascinating aspects of noncommutative scalar field theory is that when the parameter $m^2\hbar$ is sufficiently large that the potential term dominates over the kinetic term in (3.2), then the extrema of $E[\phi]$ are insensitive to the exact form of V. To be concrete, let us now assume that V has a unique global minimum, at $\phi = 0$, and a single local minimum at $\phi = \lambda$. Then every solution ϕ_0 to $V'(\phi_0)_\star = 0$ is of the form $\phi_0 = \lambda\phi$ where ϕ satisfies

$$(\phi \star \phi)(z, \bar{z}) = \phi(z, \bar{z})\,. \tag{3.3}$$

The problem of finding extrema of (3.2) therefore reduces in the $m^2\hbar \to \infty$ limit to the problem of finding projectors in the algebra $\mathcal{A}_\hbar$ of functions with respect to the star product (2.12).

Projectors may be classified by their rank, and we will begin with rank one. A class of rank one projectors is given by

$$\phi(z, \bar{z}) = f(z)\frac{C}{B(z, \bar{z})}\overline{f(z)}\,, \tag{3.4}$$

where f is an arbitrary holomorphic section, and

$$C^{-1} = c(\hbar)\int d\mu(v, \bar{v}) f(v)\overline{f(v)}e^{-\frac{1}{\hbar}K(v,\bar{v})}. \tag{3.5}$$

It is easily checked by direct substitution into (2.12) that (3.4) formally satisfies (3.3), and that its trace (2.15) is indeed 1.

It is simple to generalize (3.4) to a rank k projector by choosing k sections f_i and defining the matrix

$$h_{ij} = c(\hbar)\int d\mu(v, \bar{v}) f_i(v)\overline{f_j(v)}e^{-\frac{1}{\hbar}K(v,\bar{v})}\,. \tag{3.6}$$

Letting h^{ij} denote the inverse of h_{ij}, the desired rank k projector

$$\phi(z, \bar{z}) = \sum_{i,j} f_i(z)\frac{h^{ij}}{B(z, \bar{z})}\overline{f_j(z)} \tag{3.7}$$

satisfies (3.3) and has trace k.

Let us emphasize that the proceeding analysis has been completely general, with all formulas holding in the formal sense of deformation quantization, allowing us to construct projectors on any Kähler manifold. Of course for sufficiently complicated manifolds, such as general Calabi-Yau manifolds, it will not be possible to find explicit formulas for the quantities appearing in (3.4) because not even the metrics on these spaces are known.

In the special case when the manifold admits a quantization where functions in $\mathcal{A}_\hbar$ on $\mathcal{M}$ are represented as hermitean operators on a Hilbert space, then (3.4) is simply the symbol of the operator $\hat{\phi} = |f\rangle\langle f|/\langle f|f\rangle$. In this case it is clear that (3.4) is in fact the most general symbol of a projection operator. The symbol (3.7) corresponds to the rank k projection operator whose image is spanned by the $|f_i\rangle$.

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4. Stable solitons

When $m^2\hbar$ is large but still finite, we can analyze the effect of the kinetic term in (3.2) by doing perturbation theory in $1/(m^2\hbar)$. The space of rank one projection operators in $\mathcal{A}_\hbar$ is isomorphic to the projective space $\mathbb{P}^N$, where $N = \dim \mathcal{H}_\hbar$, and the kinetic term

$$E_1[\phi] \equiv -\lambda^2 \int d\mu(z,\bar{z})(\phi \star \triangle\phi) \tag{4.1}$$

defines a potential on $\mathbb{P}^N$. Note that the leading term in (3.2) is invariant under arbitrary unitary transformations of the Hilbert space. This $\mathrm{U}(N)$ acts transitively on the space of projection operators $\mathbb{P}^N$. The kinetic term (4.1) generically breaks this $\mathrm{U}(N)$ symmetry, and in the following sections we will discuss in several examples which subgroup (if any) of $\mathrm{U}(N)$ is preserved by the kinetic term (4.1).

To find the minima of E_1 on $\mathbb{P}^N$ we can use the method of Lagrange multipliers to enforce the condition $\phi \star \phi = \phi$. This yields the condition for the extrema

$$\phi \star \triangle\phi = \triangle\phi \star \phi\,. \tag{4.2}$$

For rank one projection operators this is equivalent to

$$(1-\phi) \star \triangle\phi \star \phi = 0\,, \tag{4.3}$$

or

$$\triangle\phi \star \phi = \alpha\phi \tag{4.4}$$

for some real number α. The minima of E_1 are given by those ϕ which satisfy (4.4) with the largest α.

It seems difficult to study (4.3) and (4.4) for general manifolds $\mathcal{M}$, but for homogeneous Kähler manifolds, as discussed in the previous section, an enormous simplification occurs because $B(z,\bar{z}) = e^{K(z,\bar{z})}$. In this case we can take a general soliton of the form (3.4) and find

$$\triangle\phi = -n\phi + g^{i\bar{j}}(\nabla_i f)e^{-K}(\nabla_{\bar{j}}\bar{f})\,, \tag{4.5}$$

where $\nabla_i \equiv \partial_i - \partial_i K$ is the covariant derivative acting on sections. If we now substitute this into (4.3) we find that on homogeneous manifolds the kinetic term (4.1) is minimized by rank one projection operators of the form $|f\rangle\langle f|$ where $f(z) = e^{K(z,\bar{v})}$ for some $\bar{v}$, i.e. $|f\rangle$ is a coherent state [24, 29]. The associated projection operator (3.4) has symbol

$$\phi_{v,\bar{v}}(z,\bar{z}) = e^{\Phi(z,\bar{z};v,\bar{v})}\,, \tag{4.6}$$

where Φ is the Calabi function as defined in (2.14), and can be thought of as a single soliton localized around $z = v$. Indeed on homogeneous manifolds the kinetic energy (4.1) for a rank one projection operator $|f\rangle\langle f|$ is proportional to the dispersion $(\Delta C)^2$ of the quadratic Casimir C of the group action in the state $|f\rangle$, and it is known [24] that the dispersion is minimized by coherent states. This will be discussed more in section 5.

Let us now consider k-soliton configurations which correspond to rank k projection operators. The space of such operators is the grassmannian $\mathrm{Gr}(k,N)$, and the kinetic energy (4.1) defines a potential on $\mathrm{Gr}(k,N)$. If we choose k points $(v_i,\bar{v}_i)$ on $\mathcal{M}$, then the projection operator whose image is spanned by the k associated coherent states $|v_i\rangle$ has symbol

$$\phi_{\{v_i,\bar{v}_i\}}(z,\bar{z}) = \sum_{ij} h_{ji} h^{ij} e^{\Phi(z,\bar{z};v_j,\bar{v}_i)}\,, \qquad h_{ij} = e^{K(v_i,\bar{v}_j)}\,. \tag{4.7}$$

It was shown in [12] that on the plane, the kinetic energy is minimized in the space of rank k projection operators precisely by those of the form (4.7) whose image is spanned by coherent states.[7] Thus there is a submanifold of the grassmannian ($\mathrm{Gr}(k,\infty)$ in that case) which is an approximate moduli space (i.e., the energy is minimized along that submanifold, but only at first order in $1/(m^2\hbar)$).

Note that throughout this paper, when we speak of a projection operator associated to k coherent states, we mean k separated coherent states. We anticipate that the expression (4.7) has a perfectly smooth limit when any two of the points are brought together (see [12] for a detailed discussion of how this works on the plane), and indeed it would be interesting to analyze in detail this merging process on a general manifold.

In the following sections we will by way of a number of examples see that on general manifolds the kinetic term is not constant on any nontrivial submanifolds of $\mathrm{Gr}(k,N)$, so that multi-soliton configurations will feel an effective force which can either bring them together or push them apart.

5. Solitons on homogeneous Kähler manifolds

In this section we apply the analysis of the preceeding sections to a number of homogeneous Kähler manifolds where explicit formulas for multi-soliton solutions are easily given. The relationship between Berezin's quantization and 'fuzzy' versions of various manifolds is also exploited.

5.1 Plane

Let us begin with the very well studied example of the plane with Kähler potential

$$K(z,\bar{z}) = z\bar{z}\,. \tag{5.1}$$

The Hilbert space is spanned by holomorphic functions of z with inner product (2.4)

$$\langle f|g\rangle = \frac{1}{2\pi}\int dz d\bar{z}\,\overline{f(z)} g(z) e^{-z\bar{z}}\,. \tag{5.2}$$

The creation and annihilation operators $a^\dagger$ and a act by differentiation and multiplication

$$(af)(z) = \frac{\partial f}{\partial z}\,, \qquad (a^\dagger f)(z) = zf(z)\,. \tag{5.3}$$

The $\star$ product (2.12) reduces to the well-known $\star$ product on the plane which corresponds to Wick ordering of the operators:

$$(O_1 \star O_2)(z,\bar{z}) = e^{\partial_v \partial_{\bar{v}}} O_1(z,\bar{v}) O_2(v,\bar{z})|_{v=z,\bar{v}=\bar{z}}\,. \tag{5.4}$$

[7] See section 7 for a discussion of the subtleties which arise when two solitons come together.

There are five different ordering of operators on the plane: pq, qp, Wick (or normal), anti-Wick, and Weyl ordering. The symbols of operators in different orderings are related by nonlocal field redefinitions. For example, the product of Weyl ordered symbols is the familiar Moyal product

$$(O_1 \star O_2)(q,p) = e^{\frac{i}{2}(\partial_{q_1}\partial_{p_2} - \partial_{q_2}\partial_{p_1})} O_1(q_1,p_1) O_2(q_2,p_2)|_{q1=q2=q;p1=p2=p} \,. \tag{5.5}$$

The symbols $O_N(z,\bar{z})$ and $O_W(z,\bar{z})$ of a given operator with respect to normal ordering and Weyl ordering respectively are related by

$$O_W(q,p) = 2\int O_N(v,\bar{v}) e^{-2(\bar{z}-\bar{v})(z-v)} dv d\bar{v} \,, \tag{5.6}$$

where

$$z = \frac{1}{\sqrt{2}}(q+ip) \,, \qquad \bar{z} = \frac{1}{\sqrt{2}}(q-ip) \,. \tag{5.7}$$

The overcomplete basis of functions $|v\rangle$ defined by (2.7) are the usual coherent states $|v\rangle = e^{\bar{v}a^\dagger}|0\rangle$. Therefore a single soliton at a position $(v,\bar{v})$ corresponds to the operator $\hat{\phi} = |v\rangle\langle v|/\langle v|v\rangle$, whose symbol is

$$\phi_{v,\bar{v}}(z,\bar{z}) = e^{-(z-v)(\bar{z}-\bar{v})} \,. \tag{5.8}$$

As already mentioned, it was shown in [12] that the kinetic term (4.1) is minimized for rank k projection operators by those whose image is spanned by coherent states. Since the kinetic term may be written in terms of operators as

$$E_1[\hat{\phi}] = \lambda^2 \,\mathrm{Tr}([a,\hat{\phi}][\hat{\phi},a^\dagger]) \tag{5.9}$$

and is therefore independent of the ordering of symbols, this result will still be true in our analysis. That is, the operator $\hat{\phi}$ which minimizes (5.9) will be the same as the one from [12], namely

$$P = \sum_{i,j} |v_i\rangle h^{ij} \langle v_j| \,, \qquad h_{ij} = \langle v_i|v_j\rangle = e^{\bar{v}_i v_j} \,, \tag{5.10}$$

but the corresponding normal-ordered symbol is

$$\phi_{\{v_i,\bar{v}_i\}}(z,\bar{z}) = \sum_{ij} h_{ji} h^{ij} e^{-(z-v_j)(\bar{z}-\bar{v}_i)} \,. \tag{5.11}$$

This corresponds to k solitons localized at the positions $(v_i,\bar{v}_i)$.

5.2 Sphere

We can introduce complex coordinates $z,\bar{z}$ on a sphere of radius R by means of the stereographic projection

$$z = R\cot(\theta/2)e^{i\varphi} \,, \qquad \bar{z} = R\cot(\theta/2)e^{-i\varphi} \,. \tag{5.12}$$

These coordinates cover the sphere except for the north pole. In these coordinates the Kähler potential is

$$K = 2R^2 \ln\left(1 + \frac{z\bar{z}}{R^2}\right) , \tag{5.13}$$

and the metric is

$$ds^2 = \frac{4}{(1 + z\bar{z}/R^2)^2} dz d\bar{z} = R^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \,. \tag{5.14}$$

Complex line bundles over the sphere are parametrized by a single integer n and denoted $\mathcal{O}(n)$. For $n \geq 0$, the bundle $\mathcal{O}(n)$ has no holomorphic sections and therefore would lead to a trivial quantum mechanics. For $n < 0$ the bundle has $N \equiv -n$ holomorphic sections, which may be written in the z-chart as the functions $1, z, \ldots, z^{N-1}$. Then the Hilbert space of geometric quantization is N dimensional.

The normalization constant c appearing in the inner product (2.4) is readily found from the formula (A.2) to be $c = (1 + 2R^2)/(4\pi R^2)$. Then (2.16) reads

$$\dim \mathcal{H}_\hbar = N = 2R^2 + 1 \,, \tag{5.15}$$

which is the familiar quantization condition on the radius of a quantum sphere.[8]

A single soliton at a position v on the sphere is described by the function

$$\phi_{v,\bar{v}}(z, \bar{z}) = \left[\frac{(1 + \frac{v\bar{z}}{R^2})(1 + \frac{z\bar{v}}{R^2})}{(1 + \frac{v\bar{v}}{R^2})(1 + \frac{z\bar{z}}{R^2})} \right]^{N-1} . \tag{5.16}$$

By taking the limit $R \to \infty$, $N \to \infty$ while keeping the relation (5.15) fixed we can go from the sphere to the infinite plane. In this limit the soliton (5.16) becomes

$$\phi_{v,\bar{v}}(z, \bar{z}) = e^{-2(z-v)(\bar{z}-\bar{v})} \,. \tag{5.17}$$

(The factor of two difference between this and (5.8) arises because in our convention (5.13) the Kähler potential on the sphere goes to $K = 2z\bar{z}$ in the flat space limit, as opposed to (5.1).)

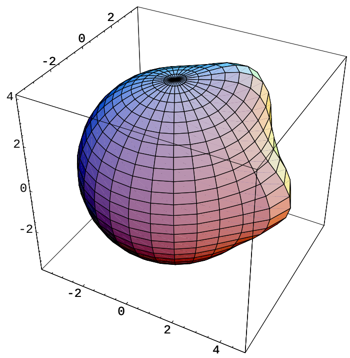

Figure 1: Two scalar solitons sitting at generic positions on a sphere of radius $R = \sqrt{29/2}$ (in units of $\hbar$), which corresponds by (5.15) to a Hilbert space of dimension $N = 30$. As discussed in section 5.2, these solitons experience an attractive force.

Using (4.7) we can easily generalize (5.16) to a function which describes k solitons at arbitrary positions $(v_i, \bar{v}_i)$ on the sphere (see figure 1 for a graphical illustration). Inserting such a function into the kinetic energy (4.1), we find an effective force which causes the two solitons to attract each other. For example, for $N = 3$ and $k = 2$, with one soliton at the south pole and the other at $z = \bar{z} = r$, the attractive potential is

$$E_1(r) = \frac{\lambda^2}{R^2} \left[2 + \frac{2}{(1 + 2R^2/r^2)^2} \right] . \tag{5.18}$$

It is instructive to see how these results can be obtained using the fuzzy sphere. Indeed one can show explicitly [30] that the $\star$ product on the sphere defined by (2.12) corresponds to the ordinary product of matrices of the fuzzy sphere, where the size of the matrix is $N = 1 + 2R^2$.

[8] Of course (5.15) is usually phrased as a quantization condition on $\hbar$. Recall that we are working in units where $\hbar = 1$, but upon restoring $\hbar$ by dimensional analysis, (5.15) reads $\hbar = 2R^2/(N-1)$.

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Let us now review some basic facts about the fuzzy sphere. The Poisson bracket algebra of the coordinate functions x_a on the sphere has the same structure as the commutation relations for SU(2) generators

$$\{x_a, x_b\} = R\epsilon_{abc}x_c\,. \tag{5.19}$$

Thus one can represent the Poisson bracket algebra of the coordinates on the sphere by associating them to matrices in the N dimensional representation of SU(2):

$$x_a \to \frac{2R}{\sqrt{N^2-1}}\hat{J}_a\,, \tag{5.20}$$

where $[\hat{J}_a, \hat{J}_b] = i\epsilon_{abc}\hat{J}_c$ and the normalization is chosen so that $x_a x_a = R^2\mathbf{1}$.

The generators of the SU(2) isometry in the z coordinates are

$$J_3 = \bar{z}\partial_{\bar{z}} - z\partial_z\,, \tag{5.21}$$

$$J_+ = -\partial_z - \bar{z}^2\partial_{\bar{z}}\,, \qquad J_- = \partial_{\bar{z}} + z^2\partial_z\,, \tag{5.22}$$

where $J_\pm \equiv J_1 \pm iJ_2$. If O is the symbol of an operator $\hat{O}$, then $J_a O$ is the symbol of $[\hat{O}, \hat{J}_a]$ The kinetic term (4.1) on the fuzzy sphere takes the form

$$E_1[\hat{\phi}] = \frac{\lambda^2}{R^2}\,\mathrm{Tr}\left(2[\hat{J}_+, \hat{\phi}][\hat{\phi}, \hat{J}_-] + [\hat{J}_3, \hat{\phi}][\hat{\phi}, \hat{J}_3]\right). \tag{5.23}$$

Written in this form it is manifest that the kinetic term breaks the U(N) invariance of the potential term in (3.2) down to an SU(2) subgroup corresponding to overall rotations of the sphere.

For a rank one projection operator of the form $P = |\psi\rangle\langle\psi|$ with $\langle\psi|\psi\rangle = 1$, (5.23) is simply

$$E_1[|\psi\rangle\langle\psi|] = \frac{2\lambda^2}{R^2}(\Delta\hat{J})^2\,, \qquad (\Delta\hat{J})^2 \equiv \langle\psi|\hat{J}_a\hat{J}_a|\psi\rangle - \langle\psi|\hat{J}_a|\psi\rangle\langle\psi|\hat{J}_a|\psi\rangle\,. \tag{5.24}$$

It is well known [24] that the dispersion $(\Delta\hat{J})^2$ is minimized when $|\psi\rangle$ is a highest (or lowest) weight state. Thus the kinetic energy is minimized by projection operators of the form $P = |v\rangle\langle v|/\langle v|v\rangle$ which project onto the coherent state

$$|v\rangle = e^{\bar{v}\hat{J}_+}|j, \pm j\rangle \tag{5.25}$$

built on a highest weight state

$$\hat{J}_3|j, \pm j\rangle = \pm j|j, \pm j\rangle\,, \qquad \hat{J}_\pm|j, \pm j\rangle = 0\,. \tag{5.26}$$

The parameter v comes from the SU(2) invariance of (5.23) which allows us to move the soliton to any point we like on the sphere. The operator P has symbol (5.16).

In the fuzzy sphere formalism it is easy to see that the (5.23) causes solitons to attract each other. One soliton sitting on the north pole would be described by $P_1 = |j, j\rangle\langle j, j|$, and two solitons sitting on top of each other by $P_2 = |j, j\rangle\langle j, j| + |j, j-1\rangle\langle j, j-1|$. It is readily verified that $E_1(P_2) < 2E_1(P_1)$, so that there is a binding energy. This is to be contrasted with the case of the plane [12], where a Bogomolny bound ensured that $E_1(P_2) = 2E_1(P_2)$. In fact it is straightforward to derive the precise potential (5.18) from (5.23) using projection operators on the fuzzy sphere.

5.3 Lobachevsky plane

We have seen that on the plane, the kinetic term (4.1) allows for multi-solitons to sit at arbitrary positions (i.e., there is a moduli space), while on the sphere, the kinetic term induces an attractive potential between solitons. In this subsection we consider the simplest homogeneous space with negative curvature, namely the Lobachevsky plane.

The Kähler potential is

$$K(z,\bar z) = -2R^2 \ln\left(1 - \frac{z\bar z}{R^2}\right), \tag{5.27}$$

where R sets the radius of curvature and $(z,\bar z)$ cover the disk inside $z\bar z < R^2$. A single soliton at position $(v,\bar v)$ is described by

$$\hat\phi_{v,\bar v}(z,\bar z) = \left[\frac{(1-\frac{v\bar v}{R^2})(1-\frac{z\bar z}{R^2})}{(1-\frac{v\bar z}{R^2})(1-\frac{z\bar v}{R^2})}\right]^{2R^2}. \tag{5.28}$$

Recall that R is measured in units of $\hbar$, which we have set to 1. Since the space of holomorphic section is infinite dimensional, there is no quantization condition, so R and $\hbar$ are both free parameters of the model.

In this case one can easily check that the kinetic term (4.1) induces a repulsive potential between two solitons. As with the sphere, it turns out to be trivial to understand this result by working with the fuzzy disk, which we now describe.

The group $\mathrm{SU}(1,1)$ has a natural action on the complex plane. Unlike the action of $\mathrm{SU}(2)$, however, the $\mathrm{SU}(1,1)$ action is not transitive but instead foliates the plane into three orbits: $X_+ = \{z : |z| < R\}$, $X_0 = \{z : |z| = R\}$, and $X_- = \{z : |z| > R\}$. Also unlike $\mathrm{SU}(2)$, which has a single type of unitary representation (labelled by spin j), the group $\mathrm{SU}(1,1)$ has three types of unitary representations: discrete, principal, and supplementary (of course all are infinite dimensional since $\mathrm{SU}(1,1)$ is noncompact), which are realized respectively on the space of functions on the orbit of type X_+, X_0, and X_-.

So for the fuzzy disk we are interested in representations in the discrete series. These are labelled by a single number k which takes discrete values, $k = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$. Basis vectors $|k,m\rangle$ are labeled by the eigenvalue of K_0,

$$K_0|k,m\rangle = (k+m)|k,m\rangle, \tag{5.29}$$

where m is a non-negative integer. The algebra is generated by operators $K_\pm$, K_0 satisfying

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_0. \tag{5.30}$$

The kinetic energy (4.1) on the fuzzy disk may be written as

$$E_1[\hat\phi] = \frac{\lambda^2}{R^2}\,\mathrm{Tr}\left(-2[K_+,\hat\phi][\hat\phi,K_-] + [K_0,\hat\phi][\hat\phi,K_0]\right) \tag{5.31}$$

and is minimized among rank one projection operators by those which project onto a coherent state

$$|v\rangle = e^{\bar v K_+}|k,0\rangle \tag{5.32}$$

built on a highest weight state.

One can check the repulsive force between two solitons by comparing the energy of a single soliton at the origin given by the projection operator $P_1 = |k,0\rangle\langle k,0|$ with the energy of two solitons sitting on top of each other at the origin, $P_2 = |k,0\rangle\langle k,0| + |k,1\rangle\langle k,1|$. It is easily verified that $E(P_2) > 2E(P_1)$, so P_2 can lower its energy by splitting apart and having the two solitons move far away from each other.

5.4 Symmetric bounded domains

According to the Cartan classification [18] there are four types of complex bounded symmetric domains $M^I_{p,q}, M^{II}_p, M^{III}_p$ and M^{IV}_n, and two exceptional domains. The elements of domains are complex matrices Z which satisfy the condition

$$ZZ^\dagger < I, \tag{5.33}$$

where Z is a complex $p \times q$ matrix for $M^I_{p,q}$, symmetric $p \times p$ for M^{II}_p and anti-symmetric $p \times p$ matrix for M^{III}_p. Here $Z^\dagger$ is hermitean conjugate and I is the identity. The fourth one M^{IV}_n is given by n-dimensional vectors. Here we will consider domains of the first three types. They are $M^I_{p,q} = \frac{\mathrm{SU}(p,q)}{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))}$, $M^{II}_p = \frac{\mathrm{Sp}(p)}{\mathrm{U}(p)}$ and $M^{III}_p = \frac{\mathrm{SO}^*(2p)}{\mathrm{U}(p)}$. In particular $M^I_{1,1} = M^{II}_1 = S^2$ and $M^I_{1,q}$ is the complex projective space $\mathbb{P}^q$.

The Kähler potential for the first three types is

$$K(Z, Z^\dagger) = \log\det(I - ZZ^\dagger)^{-\nu} \tag{5.34}$$

where $\nu = p+q, p+1, p-1$ for $M^I_{p,q}, M^{II}_p, M^{III}_p$ respectively.

We can apply the general formalism developed in sections 3 and 4 to construct solitons in these spaces. The soliton at a position $v, \bar{v}$ will be given by (4.6), where the Kähler potential is given by (5.34). Just like for the case of a sphere the solitons on a positively curved coset will attract and those on a negatively coset will repel each other.[9]

This can be checked from the group theory point of view using fuzzy cosets G/H. As we discussed with $\mathrm{SU}(1,1)$ above, the only issue is to determine which representations of G are obtained in the fuzzy coset defined by Berezin's quantization. The results are well-known [24]; for example for fuzzy $\mathbb{P}^2$ one uses $\mathrm{SU}(3)$ representations of type $(m,0)$.

6. Torus

The torus is qualitatively different from the examples we have discussed so far, because $e_\hbar(z,\bar{z}) \neq 1$, so many of the formulas remain complicated. One manifestation of this is that the $\mathrm{U}(1)\times\mathrm{U}(1)$ isometry of the torus will be 'spontaneously' broken down to a $\mathbb{Z}_N \times \mathbb{Z}_N$ subgroup generated by discrete translations by $1/N$ times a lattice vector. This symmetry breaking is a consequence of having a finite-dimensional Hilbert space.[10] The

[9] We can also imagine taking direct products of coset spaces. Then the Hilbert space will be a tensor product of two separate Hilbert spaces, and multi-solitons in different components will not talk to each other.

[10] Essentially, the problem is that translational symmetry generators $[p_x, p_y] = i$ cannot exist in a finite-dimensional Hilbert space. See [31] for a good discussion of this problem in the context of Landau levels on a torus.

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‘fuzzy’ torus studied here is therfore qualitatively quite different from the ‘noncommutative’ torus where the Hilbert space is still infinite dimensional. Noncommutative solitons on the noncommutative torus have been studied in [12, 16].

6.1 Holomorphic sections and fuzzy torus

For simplicity Let us consider a rectangular torus given by the quotient $\mathbf{T} \cong \mathbb{C}/\Gamma$, where $\Gamma = L(\mathbb{Z} + \tau\mathbb{Z})$ is a lattice on the plane and $\tau = i\tau_2$ with $\tau_2 > 0$. The constant L sets the size of the torus (in units of $\hbar$). In holomorphic gauge [31] for the magnetic potential, the appropriate Kähler potential is

$$K(z,\bar{z}) = -\frac{1}{4}(z-\bar{z})^2 = y^2\,. \tag{6.1}$$

We use the convention $z = x + iy$. The quantization condition then reads

$$\dim \mathcal{H} = N = \frac{\tau_2 L^2}{2\pi}\,. \tag{6.2}$$

In other words, the area of the torus is $\tau_2 L^2 = 2\pi N$, in agreement with the Bohr-Sommerfeld quantization rule. We choose the inner product on sections to be

$$\langle f|g\rangle = \frac{1}{N}\int_0^L dx \int_0^{\tau_2 L} dy\, \overline{f(z)} g(z)\, e^{-y^2}\,. \tag{6.3}$$

This differs by an overall constant from the conventions of section 2, but we will instead use the conventions of [27] in what follows. A basis of orthonormal sections is given by

$$f_a(z) = \frac{(2N\tau_2)^{1/4}}{\sqrt{2\pi}}\vartheta_{[a/N,0]}(zN/L,\tau N)\,, \qquad a = 0,\ldots,N-1\,, \tag{6.4}$$

where $\vartheta_{[a,b]}(z,\tau)$ is a Jacobi theta function defined by

$$\vartheta_{[a,b]}(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)}\,. \tag{6.5}$$

We will use $|a\rangle$, $a = 0,\ldots,N-1$ to denote the normalized basis vectors corresponding to the holomorphic sections f_a.

Now consider the familiar $\mathrm{SL}(2,\mathbb{Z})$ generators on the torus with $\tau = i\tau_2$,

$$(Sf)(z) = f(z+L)\,, \qquad (Tf)(z) = e^{\pi i\tau/N^2 + 2\pi iz/L} f\left(z + \frac{\tau L}{N}\right). \tag{6.6}$$

These satisfy the commutation relations

$$ST = e^{2\pi i/N}TS\,. \tag{6.7}$$

One can easily check that the action of S and T in the basis $|a\rangle$ is given by [32]

$$S|a\rangle = e^{2\pi ia/N}|a\rangle\,, \qquad T|a\rangle = |a+1\rangle\,. \tag{6.8}$$

The reader may recognize (6.8) as the fundamental algebraic relation on the fuzzy torus, which is defined as being the algebra generated by the matrices

$$U = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix}, \qquad V = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ 1 & & & \end{pmatrix}, \tag{6.9}$$

where $\omega = e^{2\pi i/N}$, which satisfy

$$VU = e^{2\pi i/N} UV \,. \tag{6.10}$$

The geometric quantization of the torus presented here is therefore precisely the fuzzy torus, and by comparing (6.9) to (6.8) we immediately see that

$$S \leftrightarrow U \,, \qquad T \leftrightarrow V^T \tag{6.11}$$

give the relation between the familiar clock and shift matrices U and V and the $\mathrm{SL}(2,\mathbb{Z})$ generators acting on sections on the torus. One can easily check that operator multiplication on the fuzzy torus in the basis we are using maps to the star product (2.12).

It is then an easy matter to write down noncommutative solitons on the torus. Let us start with a single soliton corresponding to a projection operator onto a coherent state. The Bergman kernel is

$$B(z,\bar{z}) = \frac{\sqrt{2N\tau_2}}{2\pi} \sum_{a=0}^{N-1} \left| \vartheta_{[a/N,0]}\left(\frac{zN}{L}, \tau N\right)\right|^2 . \tag{6.12}$$

The projection operator $\hat{\phi} = |v\rangle\langle v| / \langle v|v\rangle$ has symbol

$$\phi_{v,\bar{v}}(z,\bar{z}) = \frac{B(z,\bar{v})B(v,\bar{z})}{B(z,\bar{z})B(v,\bar{v})} \,. \tag{6.13}$$

This function represents a single soliton localized near $(v,\bar{v})$. One can check directly that the functional form of (6.13) is invariant only under discrete $\mathbb{Z}_N \times \mathbb{Z}_N$ translations which shift the soliton by $1/N$ times a lattice vector. Figures 2 and 3 illustrate the lack of translational symmetry graphically. We will discover in the next subsection that localized solitons of the form (6.13) do not minimize the kinetic term and are therefore unstable at first order in $1/(m^2\hbar)$.

6.2 Stable solitons

Because the translational symmetry is broken as discussed above, even defining the kinetic term (4.1) on the quantum torus is tricky. The main problem is that (4.1) as written is not well-defined, since $\triangle\phi$ is not the symbol of any bounded operator even when ϕ is, so that $\phi \star \triangle\phi$ is undefined. Another way to say this is that $\triangle\phi$ does not admit an analytic continuation to all of $\mathcal{M} \times \mathcal{M}$. As discussed at the end of section 2, this implies that the formula (2.12) holds only in the sense of deformation quantization, where $\hbar$ is treated as a formal expansion parameter.

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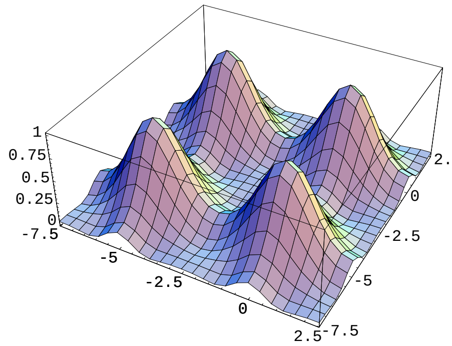

Figure 2: Four periodic copies of a single soliton on the square torus with $N = 4$ (thus $L = \sqrt{8\pi}$ in units of $\hbar$), given by (6.13) with $v = 0$.

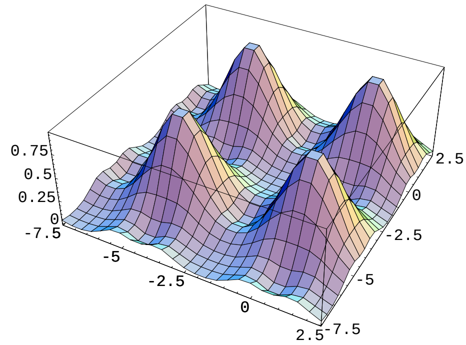

Figure 3: Same as in figure 2, but with $v = (1 + i)L/2$. The lack of translational symmetry is manifest.

One might think that since we have shown how to recast the geometric quantization of the torus into a simple algebra of $N \times N$ matrices, it would be easy to overcome these difficulties. However, they persist even on the fuzzy torus, since one would like to define the kinetic term as on the plane by (5.9), but the matrices a and $a^\dagger$ cannot exist in a finite dimensional algebra. A candidate kinetic term for the fuzzy torus has been presented in [8, 33]. In our notation, it reads

$$E_1[\hat{\phi}] = \frac{\lambda^2}{2} \operatorname{Tr}\left[(U\hat{\phi}U^\dagger - \hat{\phi})^2 + (V\hat{\phi}V^\dagger - \hat{\phi})^2\right]. \tag{6.14}$$

This kinetic term preserves the expected $\mathbb{Z}_N \times \mathbb{Z}_N$ subgroup of translational invariance,

$$E_1[T_{ab}\hat{\phi}T_{ab}^{-1}] = E_1[\hat{\phi}]\,, \qquad T_{ab} \equiv U^a V^b\,. \tag{6.15}$$

Note that (up to a phase), T_{ab} implements a discrete translation by $(a/N, b/N)$.

It turns out that the energy (6.14) is minimized only by those projection operators which either commute with U or with V — that is, the image of the projection operator must either be an eigenstate of U or an eigenstate of V. The moduli space of minima of (6.13) therefore consists of $2N$ discrete points. In position space these correspond to a strip which is localized around a $1/N$ lattice point in the x (or y) direction and extended along the y (resp. x) direction. It would be interesting to investigate the moduli space of multi-solitons in more detail using our general approach.

7. Discussion

In this paper we constructed scalar noncommutative multi-solitons on an arbitrary Kähler manifold by using Berezin's geometric approach to quantization of Kähler manifolds and its generalization to deformation quantization. For homogeneous manifolds we analyzed stability conditions for these solitons and showed that stable solitons are given in terms of generalized coherent states.

We found that on homogeneous manifolds of positive curvature, coherent state solitons tend to attract, while they repel each other on homogeneous manifolds of negative

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curvature. This is to be contrasted with the case of the plane, where the leading correction (4.1) to the energy of the solitons allows for a nontrivial moduli space. It is tempting to conjecture a general relation between curvature and the force between solitons on general manifolds. However, another important ingredient in the story is whether the Hilbert space is finite or infinite dimensional.

Indeed, it is easy to see quite generally that projection operators whose image is spanned by generalized coherent states at separated points cannot minimize the kinetic term (4.1) on a compact manifold. This is because if $\hat{\phi}$ minimizes the kinetic term in the space of rank k projectors ($\mathrm{Gr}(k, N)$), then $1 - \hat{\phi}$ minimizes the kinetic term in the space of rank $N - k$ projectors (which again is isomorphic to $\mathrm{Gr}(k, N)$) and so it too must project onto coherent states. Clearly $\hat{\phi}$ and $1 - \hat{\phi}$ should be orthogonal, but generalized coherent states are not orthogonal (in general).

It would therefore be interesting to investigate the behavior of noncommutative solitons on higher genus Riemann surfaces $\mathcal{M}_g$, which admit negative curvature metrics but are compact and therefore have a finite dimensional space of holomorphic sections. All such manifolds may be obtained as quotients of the hyperbolic plane by some discrete group. The quantization of these manifolds has been discussed in [34], where a basis of holomorphic sections was presented.

In [12] it was shown that the moduli space of noncommutative solitons on $\mathbb{C}^d$ is the Hilbert scheme of k points in $\mathbb{C}^d$. This means that the moduli space of solitons has a very interesting structure when separated solitons are brought together — for $k > 3$ and $d > 2$ the moduli space is not even a manifold! It would be interesting to see if there is a similarly rich structure when generalized coherent states of the type used in this paper come together and to explore the relation to Hilbert schemes [35], especially if there turns out to be a nontrivial moduli space for noncommutative solitons on any noncompact Ricci-flat surface.

There are many other directions for possible future investigation. In would be interesting to generalize these solitons to gauge theory and to explore their relation to D-branes on Kähler manifolds. D-branes on group manifolds have been constructed in WZW models (see [36] for review), and it would be interesting to explore the exact connection to those constructions. Also it would be interesting to compare the energy of solitons to the D-brane tensions and to derive the properties of multi-solitons (attraction or repulsion) that we obtained in section 5 from the D-brane point of view.

Finally, noncommutative solitons have a very natural interpretation in the K-theory of C^*–algebras [37, 38] and it would be interesting to understand the K-theoretic interpretation of the solitons we constructed.

Acknowledgments

We are grateful to M. Headrick, K. Hori, A. Maloney, S. Minwalla, B. Pioline, M. Wijnholt and especially to R. Gopakumar for very useful discussions. We are also indebted to A. Strominger for many useful discussions and comments on a preliminary version of the paper. This work is supported in part by DOE grant DE-FG02-91ER40654, and A.V. is also supported by INTAS-OPEN-97-1312.

A. Properties of the $\star$ product

Here we will summarize the properties of the $\star$ product, defined by (2.12). Plugging the operator $\hat{O}=1$ into (2.11) gives

$$f(z) = c(\hbar)\int f(v) e_\hbar(z,\bar{v}) e^{\frac{1}{\hbar}K(z,\bar{v})-\frac{1}{\hbar}K(v,\bar{v})} d\mu(v,\bar{v}) = \langle z|f\rangle\,; \tag{A.1}$$

from (2.12) since $1\star 1 = 1$ we have finally

$$\int \frac{e_\hbar(z,\bar{v})e_\hbar(v,\bar{z})}{e_\hbar(z,\bar{z})} e^{\frac{1}{\hbar}\Phi(z,\bar{z}|v,\bar{v})} d\mu(v,\bar{v}) = c^{-1}(\hbar)\,, \tag{A.2}$$

and from $1\star O = O\star 1 = O$

$$\begin{aligned} O(z,\bar{z}) &= c(\hbar)\int \frac{e_\hbar(z,\bar{v})e_\hbar(v,\bar{z})}{e_\hbar(z,\bar{z})} O(z,\bar{v}) e^{\frac{1}{\hbar}\Phi(z,\bar{z}|v,\bar{v})} d\mu(v,\bar{v}) \\ &= c(\hbar)\int O(v,\bar{z}) \frac{e_\hbar(z,\bar{v})e_\hbar(v,\bar{z})}{e_\hbar(z,\bar{z})} e^{\frac{1}{\hbar}\Phi(z,\bar{z}|v,\bar{v})} d\mu(v,\bar{v})\,. \end{aligned} \tag{A.3}$$

References

[1] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [hep-th/0003160].

[2] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[3] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[4] V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[5] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[6] N.A. Nekrasov, *Trieste lectures on solitons in noncommutative gauge theories*, hep-th/0011095.

[7] J.A. Harvey, *Komaba lectures on noncommutative solitons and D-branes*, hep-th/0102076.

[8] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2002) 977 [hep-th/0106048].

[9] J.A. Harvey, P. Kraus, F. Larsen and E.J. Martinec, *D-branes and strings as non-commutative solitons*, *J. High Energy Phys.* **07** (2000) 042 [hep-th/0005031]; K. Dasgupta, S. Mukhi and G. Rajesh, *Noncommutative tachyons*, *J. High Energy Phys.* **06** (2000) 022 [hep-th/0005006].

[10] A. Sen, *Descent relations among bosonic D-branes*, *Int. J. Mod. Phys.* **A 14** (1999) 4061 [hep-th/9902105]; *Universality of the tachyon potential*, *J. High Energy Phys.* **12** (1999) 027 [hep-th/9911116].

JHEP03(2002)011

[11] D. Bak and K.-M. Lee, *Elongation of moving noncommutative solitons*, *Phys. Lett.* **B 495** (2000) 231 [hep-th/0007107];
A.S. Gorsky, Y.M. Makeenko and K.G. Selivanov, *On noncommutative vacua and noncommutative solitons*, *Phys. Lett.* **B 492** (2000) 344 [hep-th/0007247];
C.-G. Zhou, *Noncommutative scalar solitons at finite theta*, hep-th/0007255;
U. Lindström, M. Roček and R. von Unge, *Non-commutative soliton scattering*, *J. High Energy Phys.* **12** (2000) 004 [hep-th/0008108];
A. Solovyov, *On noncommutative solitons*, *Mod. Phys. Lett.* **A 15** (2000) 2205 [hep-th/0008199];
B. Durhuus, T. Jonsson and R. Nest, *Noncommutative scalar solitons: existence and nonexistence*, *Phys. Lett.* **B 500** (2001) 320 [hep-th/0011139];
M.G. Jackson, *The stability of noncommutative scalar solitons*, *J. High Energy Phys.* **09** (2001) 004 [hep-th/0103217].

[12] R. Gopakumar, M. Headrick and M. Spradlin, *On noncommutative multi-solitons*, hep-th/0103256.

[13] L. Hadasz, U. Lindström, M. Roček and R. von Unge, *Noncommutative multisolitons: moduli spaces, quantization, finite theta effects and stability*, *J. High Energy Phys.* **06** (2001) 040 [hep-th/0104017].

[14] A.P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, *Phys. Lett.* **B 495** (2000) 407 [hep-th/0007043];
D.P. Jatkar, G. Mandal and S.R. Wadia, *Nielsen-Olesen vortices in noncommutative abelian Higgs model*, *J. High Energy Phys.* **09** (2000) 018 [hep-th/0007078];
D.J. Gross and N.A. Nekrasov, *Dynamics of strings in noncommutative gauge theory*, *J. High Energy Phys.* **10** (2000) 021 [hep-th/0007204];
C. Sochichiu, *Noncommutative tachyonic solitons: interaction with gauge field*, *J. High Energy Phys.* **08** (2000) 026 [hep-th/0007217];
R. Gopakumar, S. Minwalla and A. Strominger, *Symmetry restoration and tachyon condensation in open string theory*, *J. High Energy Phys.* **04** (2001) 018 [hep-th/0007226];
D. Bak, *Exact multi-vortex solutions in noncommutative abelian-Higgs theory*, *Phys. Lett.* **B 495** (2000) 251 [hep-th/0008204];
M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in noncommutative gauge theory*, *J. High Energy Phys.* **04** (2001) 001 [hep-th/0009142];
J.A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, *J. High Energy Phys.* **12** (2000) 024 [hep-th/0010060];
D.J. Gross and N.A. Nekrasov, *Solitons in noncommutative gauge theory*, *J. High Energy Phys.* **03** (2001) 044 [hep-th/0010090];
M. Hamanaka and S. Terashima, *On exact noncommutative BPS solitons*, *J. High Energy Phys.* **03** (2001) 034 [hep-th/0010221].

[15] Y. Hikida, M. Nozaki and T. Takayanagi, *Tachyon condensation on fuzzy sphere and noncommutative solitons*, *Nucl. Phys.* **B 595** (2001) 319 [hep-th/0008023];
S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, *Noncommutative gauge theory on fuzzy sphere from matrix model*, *Nucl. Phys.* **B 604** (2001) 121 [hep-th/0101102];
K. Hashimoto and K. Krasnov, *D-brane solutions in non-commutative gauge theory on fuzzy sphere*, *Phys. Rev.* **D 64** (2001) 046007 [hep-th/0101145];
Y. Kimura, *Noncommutative gauge theories on fuzzy sphere and fuzzy torus from matrix model*, *Prog. Theor. Phys.* **106** (2001) 445 [hep-th/0103192];

C.-T. Chan, C.-M. Chen, F.-L. Lin and H.S. Yang, *$\mathbb{CP}^n$ model on fuzzy sphere*, *Nucl. Phys.* **B 625** (2002) 327 [hep-th/0105087].

[16] E.M. Sahraoui and E.H. Saidi, *Solitons on compact and noncompact spaces in large noncommutativity*, *Class. and Quant. Grav.* **18** (2001) 3339 [hep-th/0012259];
I. Bars, H. Kajiura, Y. Matsuo and T. Takayanagi, *Tachyon condensation on noncommutative torus*, *Phys. Rev.* **D 63** (2001) 086001 [hep-th/0010101];
E.J. Martinec and G.W. Moore, *Noncommutative solitons on orbifolds*, hep-th/0101199;
T. Krajewski and M. Schnabl, *Exact solitons on noncommutative tori*, *J. High Energy Phys.* **08** (2001) 002 [hep-th/0104090];
H. Kajiura, Y. Matsuo and T. Takayanagi, *Exact tachyon condensation on noncommutative torus*, *J. High Energy Phys.* **06** (2001) 041 [hep-th/0104143].

[17] N. Nekrasov and A. Schwarz, *Instantons on noncommutative R^4 and $(2,0)$ superconformal six dimensional theory*, *Commun. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[18] F.A. Berezin, *Quantization*, *Izvestiya AN USSR, Ser. Math.* **38** (1974) 1116; *General concept of quantization*, *Commun. Math. Phys.* **40** (1975) 153; *Quantization in complex symmetric spaces*, *Izvestiya AN USSR, Ser. Math.* **39** (1975) 363.

[19] M. Kontsevich, *Deformation quantization of Poisson manifolds, 1*, q-alg/9709040.

[20] N. Reshetikhin and L. Takhtajan, *Deformation quantization of Kähler manifolds*, math.QA/9907171.

[21] L. Rastelli, A. Sen and B. Zwiebach, *Half strings, projectors and multiple D-branes in vacuum string field theory*, *J. High Energy Phys.* **11** (2001) 035 [hep-th/0105058];
D.J. Gross and W. Taylor, *Split string field theory, 1*, *J. High Energy Phys.* **08** (2001) 009 [hep-th/0105059].

[22] A. Volovich, *Discreteness in desitter space and quantization of kaehler manifolds*, hep-th/0101176.

[23] A. Echeverria-Enriquez, M.C. Muñoz-Lecanda, N. Roman-Roy and C. Victoria-Monge, *Mathematical foundations of geometric quantization*, *Extracta Math.* **13** (1998) 135 [math-ph/9904008].

[24] A. Perelomov, *Generalized coherent states and their applications*, Springer-Verlag, 1986.

[25] J.H. Rawnsley, *Coherent states and Kähler manifolds*, *Quart. J. Math. Oxford (2)* **28** (1977) 403;
J. Rawnsley, M. Cahen and S. Gutt, *Quantization of Kähler manifolds, 1. Geometric interpretation of Berezin's quantization*, *J. Geom. Phys.* **7** (1990) 45;
M. Schlichenmaier, *Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization*, [math.QA/9910137]; *Berezin-Toeplitz quantization and Berezin transform*, [math.QA/0009219].

[26] S. Bergman, *The kernel function and conformal mapping*, American Mathematical Society, 1970.

[27] M. Bordemann, J. Hoppe, P. Schaller and M. Schlichenmaier, $\mathrm{GL}(\infty)$ *and geometric quantization*, *Commun. Math. Phys.* **138** (1991) 209.

[28] A. Karabagedov, *Pseudo-Kähler quantization on flag manifolds*, dg-ga/9709040.

[29] M.I. Monastyrskii and A.M. Perelomov, *Coherent states and bounded homogeneous domains*, *Sov. Phys. Dokl.* **17** (1973) 12.

JHEP03(2002)011

[30] G. Alexanian, A. Pinzul and A. Stern, *Generalized coherent state approach to star products and applications to the fuzzy sphere*, *Nucl. Phys.* **B 600** (2001) 531 [hep-th/0010187].

[31] E. Onofri, *Landau levels on a torus*, *Int. J. Theor. Phys.* **40** (2001) 537 [quant-ph/0007055].

[32] D. Mumford, *Tata Lectures on Theta I*, Birkhäuser Boston, 1983.

[33] M. Chaichian, A. Demichev, P. Presnajder, M.M. Sheikh-Jabbari and A. Tureanu, *Quantum theories on noncommutative spaces with nontrivial topology: Aharonov-Bohm and Casimir effects*, *Nucl. Phys.* **B 611** (2001) 383 [hep-th/0101209].

[34] S. Klimek and A. Lesniewski, *Quantum Riemann surfaces, 2. The discrete series*, *Lett. Math. Phys.* **24** (1992) 139.

[35] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, American Mathematical Society University Lecture Series, 1999;
K. Furuuchi, *Instantons on noncommutative R^4 and projection operators*, *Prog. Theor. Phys.* **103** (2000) 1043 [hep-th/9912047];
N.A. Nekrasov, *Noncommutative instantons revisited*, hep-th/0010017.

[36] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Open strings and noncommutative geometry of branes on group manifolds*, *Mod. Phys. Lett.* **A 16** (2001) 325 [hep-th/0104054].

[37] J.A. Harvey and G.W. Moore, *Noncommutative tachyons and K-theory*, *J. Math. Phys.* **42** (2001) 2765 [hep-th/0009030].

[38] S. Fredenhagen and V. Schomerus, *Branes on group manifolds, gluon condensates and twisted K-theory*, *J. High Energy Phys.* **04** (2001) 007 [hep-th/0012164].

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